Symulacja w Badaniach i Rozwoju
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Limit cycles in systems with delay

1 A Simple Delay Model

We start from an ordinary model without delay which is commonly used to describe relaxation processes. A scalar quantity $u=u(t)$ is assumed to be in equilibrium at $u=u_0=0$. A positive deviation from the equilibrium position results in a negative rate and vice versa. Formally, we write this as

$$\dot{u}(t) = g\big(u(t)\big) \ , \tag{1}$$

where g may be in the simplest case a linear function like $g(u)=-\alpha u$. In general, we assume g to be monotonously decreasing on the whole of its domain R, which means $\alpha < 0$.

In [4, 2] the function $g(u)=-|u|u=-sign(u)u^2$ is of special interest. In economics, deviations from the true value of a commodity act in a nonlinear way on the future evolution. In fact, depending on the size of the deviation, more dealers decide that the course is too high/low, and they place higher sell/buy orders. This is assumed to exert a pressure on the value, which is proportional to the sum of all orders. Obviously, the trivial solution $u(t)=u_0=0$ of equation (1) is an asymptotically stable equilibrium state. Moreover, it is the only equilibrium, and it is globally attractive. Now, however, a second effect is added, which destabilizes the equilibrium. On the right-hand side a term is added, which yields further growth, whenever growth is detected. The point is that the detection of growth is based on the comparison of the present value with a recent value, the value of u at a delayed time $t-d$,

$$\dot{u}(t) = a(u(t)-u(t-d)) + g(u(t)) \ , \tag{2}$$

In the application to exchange rate models, dealers make decisions, whether to sell or to buy, based on their own estimate of the true value in relation to the present course. But further they assess the trend by comparing the present with a remembered course. Hence, in (2) the trend is reflected by a term proportional to the difference from the latest remembered course. In the simplest case one assumes $d = const$. Indeed, in trade one expects a regular schedule of checking the state of matters. We scale the time axis to make this value equal to *1.0* time units.

Notice that for $a = d^{-1}$, the first expression on the right-hand side is a first order approximation of the time derivative on the left-hand side. For smaller a we expect hence no qualitative change of the solution, for a =1 and larger, however, the second order contribution

$$\frac{d^2}{2} a\ddot{u}(t) \ , \tag{3}$$

begins to play a decisive role. In fact, using Taylor formula of second order as an approximation of the delayed value $u(t-d)=u(t-1)\approx u(t)-\dot{u}(t)+\frac{1}{2}\ddot{u}(t)$, the behavior is now more like that of the equation

$$0=-\frac{a}{2}\ddot{u}(t)+(a-1)\dot{u}(t)+g(u(t)) \tag{4}$$

For $a = d = 1$ and in the case of a linear function $g(u) = -u$ this will be

$$0=\ddot{u}(t)+2u(t)\ , \tag{5}$$

which leads to harmonic oscillations, while for $a > 1$ the amplitude will grow exponentially, starting from any small vicinity of $u_0 = 0$. Hence, in fact, for $a \geq 1$ the zero solution becomes unstable. For over-linear functions g, however, the growth is limited, and solutions finally tend to a limit cycle. In more advanced models, the delay $d \geq 0$ is allowed to be variable in the form

$$d=d(u)\ , \tag{6}$$

with the only assumption of scaling, continuity and monotonicity, i.e., farther deviations from equilibrium result in shorter intervals, in which the course is checked,

$$d(0)=1\ , \tag{7}$$

$$0\leq u_1\leq u_2 \Rightarrow d(0)\geq d(u_1)\geq d(u_2)\geq 0\ , \tag{8}$$

$$0\geq u_1\geq u_2 \Rightarrow d(0)\geq d(u_1)\geq d(u_2)\geq 0 \tag{9}$$

Now, the form of the evolution equation of the dynamical delay model is

$$\dot{u}(t)=a(u(t)-u(t-d(u(t))))+g(u(t))\ , \tag{10}$$

where we deliberately did not divide the difference term by the delay. This way we reflect the ‘psychological’ element of the traders’ activity. Whenever larger deviations from the expected (zero) value of the course appear, dealers become (in their average) more and more nervous and look in shorter and shorter intervals for changes. In an academic version, this is reflected by the function

$$d(u)=\exp(-cu^2)\ , \tag{11}$$

Of course, a nonsymmetric behavior for positive/negative deviations could be expected as well. Analytical studies so far are restricted to a dependence of the delay on $|u|$, further the functions g and d have to be such that $t - d(u(t))$ be always increasing. Sufficient conditions have been given in [1]. There it is also shown that for $a \leq 1$ the trivial solution is stable. For $a > 1$ the equilibrium becomes unstable, a stable limit cycle appears. In this paper, however, we will not study the analytical properties of models like (10). Instead, we will calculate numerical solutions to initial value problems and study their behavior. In particular, we will show the difference between the constant and variable delay cases for some chosen delay functions. The numerical approach is not limited by the strong conditions imposed in [1]. The appearance of attracting limit cycles is independent of d being constant or not. The size and shape of cycles, however, turns out to differ considerably for both cases.

2 Simulation

In models with delay the state of the system is represented by at least a certain segment of the history of the considered quantity. The state space is hence not finite dimensional, even if the temporal evolution of a single scalar value is studied. As a consequence, initial condition and the state of the system are represented in computer simulations as a vector of nodal values of the recent history of the studied quantity. In the present case, the delay is bounded by its value for the trivial state, which after scaling is set to be equal to *1*. Given a model with constant delay, it is possible to use a constant sampling rate, which is an integer fraction of the delay, and thus only nodal values of *u* are needed to evaluate the rate of change $\dot{u}(t)$. At each time step, the actual value of *u* is updated according to its rate of change, while the history is updated by a simple index shift. Thus the history runs through its domain like a wave in a convection problem. In the variable delay case, however, interpolation between nodal points becomes necessary. For the present application, we implemented a linear spline approximation for the evaluation of the rate. The calculation of the next value of u, i.e. for the approximation of $u(t + \Delta t)$, we implemented several explicit methods, like Euler, Euler-Heun, classical Runge-Kutta and Dormand-Prince methods. With stepsizes like $\Delta t = 0.001$, all mentioned procedures give acceptable results. However, the higher order methods of Runge-Kutta type give a much better resolution of the details in phases of sudden change, such as e.g. in Fig. 5. This is decisive in the variable delay case. For many reasons, amongst others to avoid licence problems and to allow GUI controlled animation, we chose java as the environment for the implementation. A screenshot of an animation is shown in Fig. 1. For comparison and verification, we solved identical initial-value problems using matlab's *ddesd* routine for differential systems with delay. Results were the same up to the assumed error margin. We remark that, in general, a simulation tool has to serve several purposes.

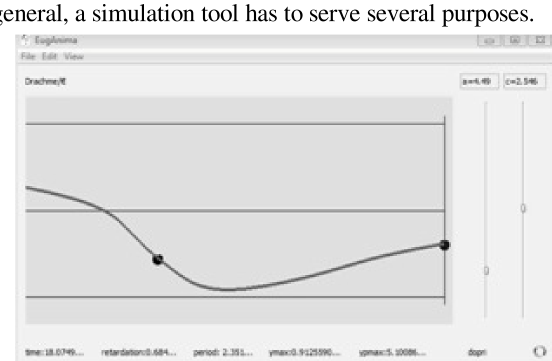

Fig. 1. Screenshot of java simulation tool. On the horizontal axis the last time unit is presented, the marker at the right end shows the present state, the other one indicates the delay and the delayed value.

Rys. 1. Zrzut z ekranu z aplikacji javy. Na osi odciętych przedstawiono ostatni jednostkowy przedział czasu, znak na prawym końcu wskazuje stan aktualny, drugi zaś pokazuje opóźnienie i stan poprzedni.

One wants an interactive GUI control over animation, and at the same time a systematic manipulation of model parameters in order to perform studies of parameter sensitivity. Preferably, the first requirement should not be delayed by the second. In the simulation tool, we used different threads in order to split the numerical load between several processor cores. It is possible and desirable to assign the systematic parameter studies to several servers, while performing the actual animation on the local host.

3 Results

In this section we show a selection of simulated exchange rates, based on the dynamical delay model (10). We show a temporal trajectory, starting from a given history, in Fig. 2. The nontrivial cyclic behavior seen in Fig. 2 can be observed for values of the model parameter a starting from $a = 1$. In Fig. 3 approximated values of the amplitudes have been plotted against the value of a. Technically, a trajectory starting from a perturbed state near equilibrium has been traced over 10 cycles, the maximum of u over the next cycle has been assumed as amplitude of the limit cycle. We observe the onset of instability at around $a = 0.99$. Further, the frequency of the limit cycle is studied, again for varying a, but for one fixed delay function d and the standard nonlinear function g.

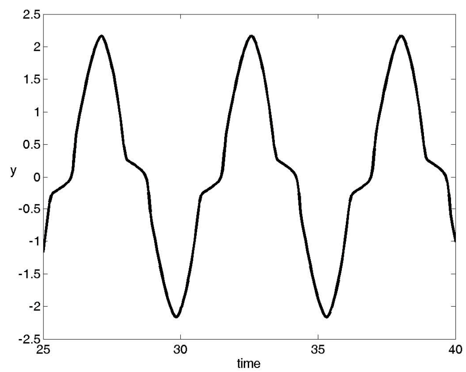

Fig. 2. Course values as function of time. After a transient phase the solution becomes periodic. The period is always longer than the maximum delay.

Rys. 2. Wartości kursu jako funkcja czasu. Po fazie przejściowej rozwiązanie staje się okresowe. Okres zawsze jest dłuższy od maksymalnego opóźnienia.

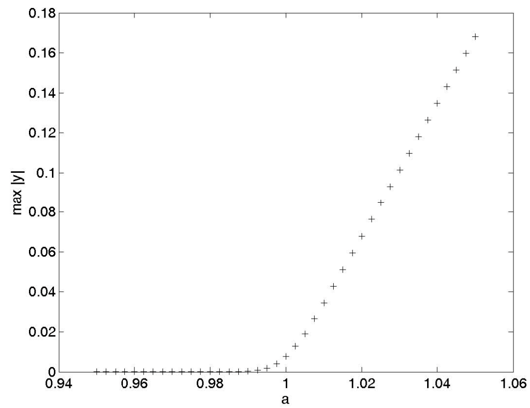

Fig. 3: Amplitudes of limit cycle as function of a. The onset of instability is at a ≈ 1.
Rys. 3: Amplitudy cykli w zależności od a. Utrata stabilności następuje przy a ≈ 1.

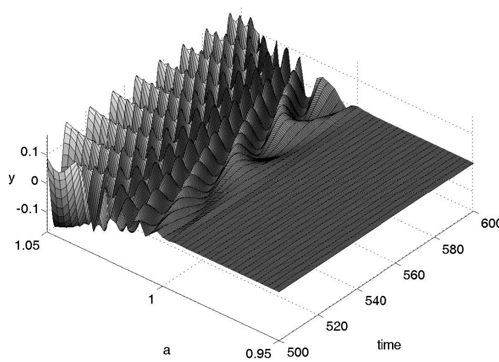

Fig. 4. Study of the frequency in dependence on parameter a. The period decreases from around 30 to 10, when a increases from 1 to 1.05

Rys. 4. Częstość cykli w zależności od a. Okres maleje od około 30 do 10, kiedy a rośnie od 1 do 1.05.

It is evident, see Fig. 4, that the frequency is increasing, the larger *a*, the larger the amplitude, and the shorter the period of the cycles. The results presented in Figs. 3 and 4 have been obtained using background threads, as mentioned in the previous section. Next, we studied the sensitivity of the model to the choice of the delay function *d*.

The effect of shrinking delay in states far from equilibrium results in reduced cycles, presented as phase portraits in Fig. 5. Note the non-convex shape and the sharp curvatures of the cycles in the variable delay case. The delay itself reaches its maximum when *u* vanishes, notice that there are phases of slower change near the equilibrium. Then, there is a quick drop of the delay to approximately *1/3* and an equally quick recovery, cf. Fig. 6.

For analysis and numerical implementation, the rate of the delayed time $t-d(u(t))$ plays an essential role. In fact, the analysis in [1] relays on this quantity to be strictly increasing. Fig. 7 shows that this condition is not necessary. In fact, the delayed time may run backwards on short intervals. For the implementation this means that the history of the studied quantity must not be forgotten once it has been used. In fact, the same past state value maybe used more than once for comparison with future values.

4 Conclusions

The considered delay model describing the dynamics of a single scalar quantity yields self-generated cycles in a certain range of model parameters, while for small parameters the trivial equilibrium solution is stable. Computer simulations are qualitatively in accordance with analytical results if the latter are applicable. Further, two different numerical methods were verified to give consistent results. A java simulation tool with graphical user interface allows to perform fast parameter studies and to quickly assess the effect of changes in model components.

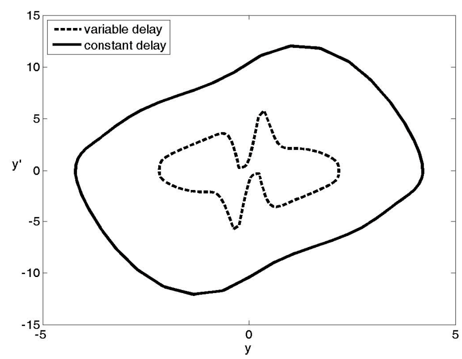

Fig. 5. Limit cycles for constant and variable delay on the speed vs value plane. Typically in the variable delay case, the transition through the equilibrium point leads to sharp changes of direction. Maximum deviations and speeds are smaller than for a constant delay of d = 1.

Rys. 5. Cykle graniczne w przypadku stałego i zmiennego opóźnienia w płaszczyźnie aktualnej szybkości i wartości.
Typowe jest, ze przy zmiennym opóźnieniu przejście przez punkt równowagi wiąże się ze znaczna zmianą kierunku. Maksymalne odchylenia kursu i szybkości są mniejsze niż przy stałym opóźnieniu równym 1.

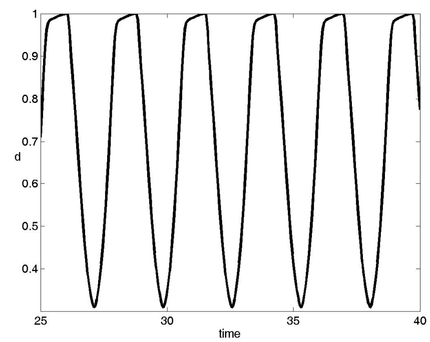

Fig. 6. The delay as a function of time.
When approaching the equilibrium point, the delay increases, so that the retarded time hardly grows.

Rys. 6. Opóźnienie jako funkcja czasu. Przy zbliżeniu do punktu równowagi wartość opóźnienia rośnie, tak że może dochodzić do stagnacji czasu opóźnionego.

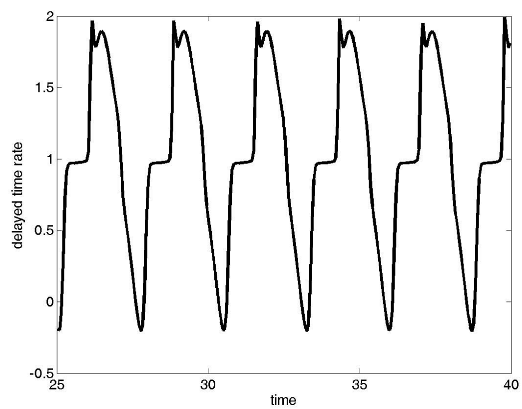

Fig. 7. The rate of the retarded time as function of present time. At certain moments the delayed time may run backwards – if the delay grows faster than the present time runs forward.

Rys. 7. Szybkość czasu opóźnionego jako funkcja czasu aktualnego.
W pewnych chwilach czas opóźniony może biec wstecz. Zdarza się to, gdy opóźnienie rośnie szybciej niż biegnie aktualny czas.

It turned out that the model behavior is robust with respect to the choice of the nonlinearity g and the delay function d. In future work the model should be refined and extended. The vector case, were more than one currency is traded, several delays and nonsymmetric delay functions may occur, is still open for analysis. The simulation tool developed for the currency exchange rate model may be extended to other applications such as for instance the air flow control in Otto engines.

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Summary

Computer simulation of real systems requires often to take into account values of crucial quantities not only at the present time, but also their recent history. Models allowing for a dependence of the rate of change on delayed values of the modeled quantity have been applied in economics to explain cycles in the exchange rates of foreign currencies. For a certain type of model equations, the existence of stable limit cycles has been shown analytically. Certain restrictive conditions on the constitutive functions, describing the behavior of traders, had to be assumed. In this paper, the analytical results are confirmed by computer simulation. Moreover, the existence of limit cycles is computationally verified under much weaker assumptions.

Cykle graniczne w układach z opóźnieniem

Streszczenie

Symulacja komputerowa systemów rzeczywistych częstokroć wymaga uwzględnienia wartości istotnych wielkości nie tylko w chwili aktualnej, a także w chwilach przeszłych. Modele wykazujące zależność szybkości zmiany badanej wielkości od historii przebiegu zostały wprowadzone w ekonomii, żeby wyjaśnić cykliczne zmiany kursów walut. W przypadku pewnego typu równań modelowych istnienie stabilnych cykli granicznych zostało udowodnione analitycznie. W tym celu na funkcje opisujące zachowanie maklerów musiały zostać narzucone pewne ograniczające warunki. W pracy weryfikowano wyniki analityczne drogą symulacji komputerowej. Ponadto pokazano obliczeniowo istnienie cykli granicznych przy znacznie słabszych.